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# The critical behaviour of self-dual $Z(N)$ spin systems: finite-size scaling and conformal invariance

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**Abstract.** This paper is concerned with the critical properties of a family of self-dual two-dimensional $Z(N)$ models whose bulk free energy is exactly known at the self-dual point. Our analysis is performed by studying the finite-size behaviour of the corresponding one-dimensional quantum Hamiltonians which also possess an exact solution at their self-dual point. By exploring finite-size scaling ideas and the conformal invariance of the critical infinite system we calculate, for $N$ up to 8, the critical temperature and critical exponents as well as the central charge associated with the underlying conformal algebra. Our results strongly suggest that the recently constructed $Z(N)$ quantum field theory of Zamolodchikov and Fateev is the underlying field theory associated with these statistical mechanical systems. We also test, for the $Z(5)$ case, the conjecture that these models correspond to the bifurcation points in the phase diagram of the general $Z(N)$ spin model, where a massless phase originates.

## 1. Introduction

In the past few years two-dimensional $Z(N)$ spin systems have been intensively studied firstly because they are interesting non-trivial systems in their own right (Domany and Riedel 1979, Alcaraz and Köberle 1980, Cardy 1980) and secondly because they share many similar properties with four-dimensional $Z(N)$ gauge systems (Fradkin and Susskind 1978, Elitzur *et al* 1979, Kogut 1979, Creutz *et al* 1979, Alcaraz and Köberle 1981). These spin models are self-dual and for $N \geqslant 5$ they exhibit the massless phase precursor of the disordered low-temperature phase of the planar $XY$ model.

Fateev and Zamolodchikov (1982) by looking for possible solutions of the star-triangle relations for $Z(N)$ models were able to find the free energy per particle for a family of self-dual points in the parameter space of the models on the square lattice. The questions that promptly arise are: are those points critical ones? If they are critical what is the universality class governing their critical behaviour and what is the underlying quantum field theory describing their criticality? In this paper we study these questions by performing a finite-size analysis on these models.

More recently Zamolodchikov and Fateev (1985) have constructed $Z(N)$-invariant quantum field theories in $(1+1)$ dimensions that are the natural candidates for the underlying field theories associated with the above statistical models. These theories are self-dual and conformally invariant with their conformal anomaly, or central charge

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of their Vivasoro algebra, given by

$$c = 2(N-1)/(N+2). \tag{1.1}$$

They have $(N-1)$ fields (order parameters) with anomalous dimensions

$$2d_n = n(N-n)/N(N+2) \qquad n = 1, 2, \ldots, N-1 \tag{1.2}$$

and $(N-1)$ dual fields (disorder parameters) with the same dimensions as in (1.2) due to the self-dual behaviour of the theory. There are also $Z(N)$ neutral fields with dimension

$$2D_n = 2n(n+1)/(N+2) \qquad n = 1, 2, \ldots, \bar{N} \tag{1.3}$$

where $\bar{N}$ is the integer part of $N/2$. If these quantum field theories actually describe the above two-dimensional statistical models the relations (1.2) and (1.3) should give the $Z(N)$ charged ('magnetic') and neutral ('thermal') exponents of the statistical models. It is interesting to remark that the above dimensions (1.2) and (1.3) correspond exactly to the exponents of the antiferromagnetic critical points of the RSOS model (Andrews *et al* 1984, Huse 1984).

In this paper by using finite-size scaling (Barber 1983) and exploring the consequences, on a finite lattice, of the conformal symmetry of the infinite critical system (Cardy 1987) we will verify that the relations (1.1)-(1.3) hold for $N<9$. A short account of these results for $N=5$ has already been presented (Alcaraz 1986).

It has also been conjectured that the above exactly soluble points (Fateev and Zamolodchikov 1982) correspond, for $N \geqslant 5$, to the bifurcation points of a general self-dual $Z(N)$ model (Alcaraz and Köberle 1980, 1981) where a soft phase appears. We also try to verify this conjecture for the case $N=5$.

The layout of this paper is as follows. In § 2 we introduce the general self-dual $Z(N)$ model while in §3 we present the family of exactly soluble points of Fateev and Zamolodchikov (1982) as well as its corresponding quantum Hamiltonian. In §§ 4 and 5 in order to test the predictions (1.1)-(1.3) we calculate the critical temperature, critical exponents and conformal anomaly associated with these models. The conjecture that these exactly soluble points are the bifurcation points in the phase diagram of a general $Z(N)$ self-dual model is analysed in § 6. Lastly § 7 consists of an overall summary and conclusion of the results presented in this paper.

## 2. The general self-dual $Z(N)$ model

The spin models we are concerned with in this paper are defined in terms of $Z(N)$ spin variables

$$S(\boldsymbol{r}) = \exp \frac{\mathrm{i}2\pi}{N} n(\boldsymbol{r}) \qquad (n(\boldsymbol{r}) = 0, 1, \ldots, N-1) \tag{2.1}$$

located at the sites $\boldsymbol{r} \equiv (i, j)$ of a square lattice. The most general self-dual $Z(N)$ spin model with only nearest-neighbour interactions, on the square lattice, is defined by the reduced Hamiltonian

$$H = \sum_{ij} [H_1(n(i,j) - n(i+1,j)) + H_{-1}(n(i,j) - n(i,j+1))] \tag{2.2a}$$

where

$$H_k(n) = -\sum_{m=1}^{\bar{N}} J_m^{(k)} \left[\cos\left(\frac{2\pi}{N} mn\right) - 1\right] \qquad k = -1, 1 \tag{2.2b}$$

and as before $\bar{N}$ is the integer part of $N/2$ and $J_m^{(k)}$; $k=-1,1$, $m=1,2,\ldots,\bar{N}$ are the coupling constants in the horizontal and vertical directions, respectively. The cases $N=2$ and 3 correspond to the Ising and three-state Potts model while $N=4$ corresponds to the symmetric Ashkin–Teller model (Ashkin and Teller 1943). There are $N$ Boltzmann weights associated with each direction in the lattice

$$X_n^{(k)} \equiv \exp(-\beta H_k(n)) \qquad k=-1,1 \qquad n=0,1,\ldots,N-1. \tag{2.3}$$

Only $2\bar{N}$ of these are distinct due to the fact that

$$H_k(n)=H_k(N-n) \qquad k=-1,1 \qquad n=0,1,\ldots,N-1. \tag{2.4}$$

The family of models

$$X_n^{(k)}=X^{(k)} \qquad k=-1,1 \qquad n=1,2,\ldots,N-1 \tag{2.5}$$

correspond to the $N$-state Potts models (Potts 1952) while

$$J_n^{(k)}=J^{(k)}\delta_{n,1} \qquad k=-1,1 \qquad n=1,2,\ldots,N-1 \tag{2.6}$$

correspond to the $N$-state clock model (José *et al* 1977, Elitzur *et al* 1979). Under the duality transformation the Boltzmann weights (2.3) are transformed to (Alcaraz and Köberle 1980, 1981, Cardy 1980)

$$\tilde{X}_n^{(k)}=\left[\sum_{m=0}^{N-1}\exp\left(\frac{\mathrm{i}2\pi mn}{N}\right)X_m^{(-k)}\right]\left(\sum_{m=0}^{N-1}X_m^{(-k)}\right)^{-1} \qquad k=-1,1. \tag{2.7}$$

The self-dual subspace, fixed under the duality transformation

$$\tilde{X}_n^{(k)}=X_n^{(k)} \qquad k=-1,1 \qquad n=0,1,\ldots,N-1 \tag{2.8}$$

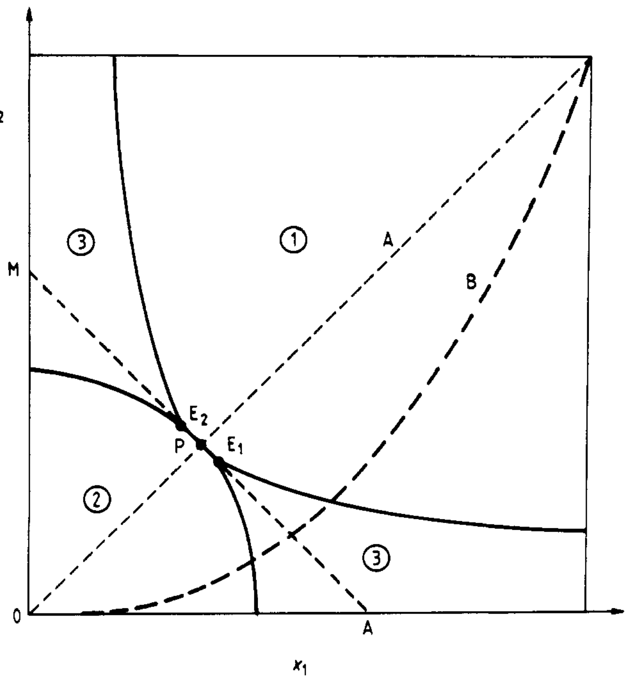


**Figure 1.** Schematic phase diagram of the general isotropic $Z(5)$ model (see text).

is a line for $N=2, 3$, a plane for $N=4, 5$, etc, and coincides with the critical surface in the regions of the parameter space in which the transition is unique.

The general features of the phase diagram of the Hamiltonian (2.2) are reasonably well understood (Alcaraz and Köberle 1980, 1981, Cardy 1980). For $N\leqslant 4$ the transitions are all continuous and all phases are massive (finite correlation length). For $N\geqslant 5$ first-order transitions are found (for example the Potts model) and a massless phase (infinite correlation length) appears in the phase diagram. In order to illustrate this we show in figure 1 the phase diagram for the isotropic $Z(5)$ model: $X_1^{(1)}=X_1^{(-1)}=X_1$; $X_2^{(1)}=X_2^{(-1)}=X_2$ (Alcaraz and Köberle 1980, 1981). The phases 1 and 2 are the paramagnetic and ferromagnetic phases while 3 represents the massless phases. The straight line AM is the self-dual line. The critical point of the five-state Potts model is P while $\mathrm{E}_1$, $\mathrm{E}_2$ are the bifurcation points where the massless phases originate. The straight line A and the curve B are the thermodynamical paths of the five-state Potts model and five-state clock model, respectively.

## 3. Fateev–Zamolodchikov solution and the associated quantum Hamiltonian

Fateev and Zamolodchikov (1982), by restricting the general model (2.2) to the self-dual subspace, were able to solve the star–triangle relations and calculate the free energy per particle for a particular set of isolated points in the parameter space. Their solution corresponds to the Boltzmann weights

$$\begin{aligned} &X_0^{(k)}=1 \qquad X_n^{(1)}=f_n(\alpha) \qquad X_n^{(-1)}=f_n(\pi-\alpha) \\ &k=-1,1 \qquad n=1,\ldots,N-1 \end{aligned} \tag{3.1a}$$

where

$$f_n(\alpha)=\prod_{k=0}^{n-1}\sin\left(\frac{\pi k}{N}+\frac{\alpha}{2N}\right)\left[\sin\left(\frac{\pi(k+1)}{N}-\frac{\alpha}{2N}\right)\right]^{-1} \tag{3.1b}$$

and $\alpha$ is an arbitrary constant that fixes the anisotropy of the model (for the isotropic system $\alpha=\pi/2$). For fixed $N$ the weights (3.1) correspond to a point in the parameter space between the thermodynamical path of the $N$-Potts (2.5) and $N$-clock (2.6) models. On the other hand, for $N\geqslant 5$ the phase diagram of the general model (2.2) also has a special point between these two thermodynamical paths, namely the bifurcation point where the massless phase originates, corresponding to the point $\mathrm{E}_1$ in figure 1 (Alcaraz and Köberle 1980, 1981). This suggests the conjecture (Fateev and Zamolodchikov 1982) that the special family (3.1) corresponds to these bifurcation points in the phase diagram. In § 6 we will test this conjecture for the $N=5$ model.

Recently by looking for Lax pair solutions associated with quantum Hamiltonians it has been shown (Alcaraz and Lima Santos 1986) that the family of one-dimensional quantum $Z(N)$ models whose dynamics is governed by the Hamiltonian

$$H_N=-\sum_{i=-\infty}^{\infty}\sum_{n=1}^{N-1}[S^n(i)S^{+n}(i+1)+R^n(i)]/\sin(\pi n/N) \tag{3.2}$$

has an infinite number of local and non-local conservation laws. In (3.2) $S(i)$ and $R(i)$ are quantum operators satisfying the $Z(N)$ algebra (Alcaraz and Köberle 1980, 1981)

$$[S(i),R(j)]=[S(i),S(j)]=[R(i),R(j)]=0 \qquad i\neq j \tag{3.3a}$$

$$S(i)R(i)=\exp(\mathrm{i}2\pi/N)R(i)S(i) \qquad R^N(i)=S^N(i)=1. \tag{3.3b}$$

Moreover the generator of the infinite set of charges corresponds to the diagonal-to-diagonal transfer matrix $T_D$ of the general classical model (2.2) at the exactly soluble point (3.1), the first conserved charge being the Hamiltonian (3.2), i.e.

$$[T_D, H_N] = 0. \tag{3.4}$$

The above relation implies that we can, in an equivalent way, study the Hamiltonians (3.2) instead of the Euclidean model (2.2) at the couplings (3.1). From the computational point of view this represents a great simplification because the Hamiltonian (3.2) is a sparse matrix while the transfer matrix associated with (3.1) is very dense. Fortunately the ground-state energy per particle of (3.2), for the infinite system, is also known exactly (Alcaraz and Lima Santos 1986)

$$E_0 = -N \int_0^\infty \frac{\sinh(\frac{1}{2}\pi x)\sinh[\frac{1}{2}\pi x(N-1)]}{\cosh^2(\frac{1}{2}\pi x)\cosh(\frac{1}{2}\pi N x)}\,dx - \sum_{n=1}^{N-1} \frac{1}{\sin(\pi n/N)}. \tag{3.5}$$

In table 1 we present the numerical values for $N \leq 8$.

## 4. Critical temperature and exponents

The purpose of this section is firstly to verify that the family of exactly soluble points (3.1) of the general model (2.2) are critical points and secondly to calculate their critical exponents. This will enable us to test if the $Z(N)$ quantum field theory recently constructed by Zamolodchikov and Fateev (1985) corresponds to the underlying field theory of these statistical models. According to this field theory the 'magnetic' ($Z(N)$ charged) and 'thermal' ($Z(N)$ neutral) critical indices are given by (1.2) and (1.3) respectively.

**Table 1.** Exact ground-state energy per particle of Hamiltonian (3.2).

| $N$ | $e_0$ |
|---|---|
| 2 | −1.273 239 544 ... |
| 3 | −2.812 840 265 ... |
| 4 | −4.546 479 089 ... |
| 5 | −6.431 029 721 ... |
| 6 | −8.438 520 787 ... |
| 7 | −10.549 521 776 ... |
| 8 | −12.749 812 427 ... |

For $N = 2$ and $N = 3$ the relations (1.2) and (1.3) are easily verified because in this case the $Z(N)$ model with Boltzmann weights given by (3.1) corresponds to the critical Ising and critical three-state Potts model respectively. For the Ising (three-state Potts) model the anomalous dimension corresponding to the energy operator is $X_\varepsilon = 1$ ($X_\varepsilon = \frac{4}{5}$) and to the magnetic operator is $X_m = \frac{1}{8}$ ($X_m = \frac{2}{15}$) which fully agree with the predictions (1.2)–(1.3) (den Nijs 1979, Belavin *et al* 1984a, b).

Hereafter we will exploit the relation (3.4) to simplify our numerical calculations. Instead of studying the Euclidean version of the models given by (2.2) and (3.1) we concentrate on its Hamiltonian version given by (3.2). In order to test whether the Hamiltonians (3.2) are critical we should extend them by introducing a coupling $\lambda$ to play the role of temperature

$$H_N(\lambda) = -\sum_{i=-\infty}^{\infty} \sum_{n=1}^{N-1} [[\lambda R^n(i) + S^n(i) S^{+n}(i+1)]/\sin(\pi n/N)]. \tag{4.1}$$

This Hamiltonian is self-dual $H_N(\lambda) = \lambda H_N(1/\lambda)$ and at its self-dual point ($\lambda = 1$) it reduces to (3.2). It can also be obtained in an appropriate time continuum limit ($\alpha \to 0$) (Fradkin and Susskind 1978) of (2.2) around the point (3.1).

The $Z(4)$ case can be better analysed by writing the operators $S(i)$, $R(i)$ in (3.3) in terms of two Pauli matrices $\sigma^z(i)$, $\sigma^x(i)$; $\tau^z(i)$, $\tau^x(i)$ located at each lattice point

$$S(i) = (1/\sqrt{2})(e^{i\pi/4}\sigma^z(i) \oplus \mathbb{1} + e^{-i\pi/4}\mathbb{1} \oplus \tau^z(i)) \qquad S^2(i) = \sigma^z(i) \oplus \tau^z(i) \tag{4.2a}$$

$$R(i) + R^+(i) = \sigma^x(i) \oplus \mathbb{1} + \mathbb{1} \oplus \tau^x(i) \qquad R^2(i) = \sigma^x(i) \oplus \tau^x(i). \tag{4.2b}$$

In terms of these Pauli matrices the Hamiltonian (4.1) for $N = 4$ is given by

$$\begin{aligned} H_4 = -\sum_i \{ & \lambda[\sqrt{2}(\sigma^x(i) + \tau^x(i)) + \sigma^x(i)\tau^x(i)] \\ & + [\sqrt{2}(\sigma^z(i)\sigma^z(i+1) + \tau^z(i)\tau^z(i+1)) \\ & + \sigma^z(i)\sigma^z(i)\sigma^z(i+1)\tau^z(i)\tau^z(i+1)]\}. \end{aligned} \tag{4.3}$$

This Hamiltonian corresponds to a particular case $\beta = \sqrt{2}/2$ (see Kohmoto *et al* 1981, Alcaraz and Drugowich de Felício 1984) of the quantum Ashkin–Teller model. This model is critical at $\lambda = 1$ with the exponents

$$\nu = \tfrac{3}{4} \qquad \gamma_m = \tfrac{21}{16} \qquad \text{and} \qquad \gamma_p = \tfrac{5}{4}$$

for the correlation length, magnetisation and polarisation, respectively. These exponents give the dimension $X_\varepsilon = d - 1/\nu = \frac{2}{3}$ for the energy operator while for the magnetic and polarisation operators they give the dimensions $X_m = (d - \gamma_m/\nu)/2 = \frac{1}{8}$ and $X_p = (d - \gamma_p/\nu)/2 = \frac{1}{6}$ respectively, which completely agree with the predictions (1.2) and (1.3). It is interesting to observe that the second $Z(4)$ neutral operator whose dimension is predicted by (1.3) is marginal ($X_{\varepsilon\varepsilon} = 2$) and probably corresponds to the well known marginal operator (four-spin coupling) of the eight-vertex and Ashkin–Teller (Kadanoff and Wegner 1971, Kadanoff and Brown 1979) models.

For $N > 4$ all our results will be obtained by studying the behaviour of finite lattices of size $L$ as $L$ goes to infinity.

### 4.1. Finite-size scaling (FSS)

The Hamiltonians (4.1) for a finite lattice of size $L$ and periodic boundary conditions commutes with the $Z(N)$ charge operator

$$\exp\left(\frac{i2\pi}{N} Q\right) = \prod_{i=1}^{L} R(i). \tag{4.4}$$

In the basis where the $R(i)$ operators are diagonal the Hilbert space is then separated into $N$ disjoint sectors labelled by the eigenvalues of $Q(q = 0, 1, \ldots, N-1)$. The ground state is in the $Q = 0$ sector while the sectors with $Q = q$ and $Q = N - q(q = 1, 2, \ldots, N-1)$ are degenerate. These sectors can be further block diagonalised according to the eigenvalues of the translation operator (linear momentum). From the lowest eigenenergies $E_0^{(q)}(\lambda, L)$ of the sector $q(1, 2, \ldots, N-1)$ we can define $N - 1$ mass gaps

$$\Lambda_L^{(q)} \equiv E_0^{(q)}(\lambda, L) - E_0^{(0)}(\lambda, L) \qquad q = 1, 2, \ldots, N-1. \tag{4.5}$$

Due to the degeneracy of the Hilbert space only the first $\bar{N}$ of these are distinct. All eigenvalue calculations throughout this paper were performed by using the Lanczos method (Hamer and Barber 1981a, Roomany *et al* 1980) starting with an appropriate state for each disjoint sector of the Hilbert space.

According to finite-size scaling theory (FSS) (Barber 1983) the critical temperature $\lambda_c$ may be estimated by the extrapolation to the bulk limit ($L \to \infty$) of the sequences $\lambda_c^{(q)}(L)$, $q = 1, 2, \ldots, \bar{N}$ obtained by solving the equations

$$L\Lambda_L^{(q)}(\lambda_c^{(q)})/(L-1)\Lambda_{L-1}^{(q)}(\lambda_c^{(q)}) = 1 \qquad q = 1, 2, \ldots, \bar{N}. \tag{4.6}$$

In table 2 we exhibit these sequences for $N = 5$ ($L = 3$-$8$), $N = 6$($L = 3$-$7$), $N = 7$ ($L = 3$-$6$) and $N = 8$ ($L = 3$-$6$). These tables are shorter for $N = 7, 8$ because as $N$ grows the dimension of the Hilbert space grows proportional to $N^L$ increasing the computational difficulty in diagonalising (4.1) for larger lattices.

Due to the self-duality of the Hamiltonian (4.1) three distinct possible critical behaviours may occur: (i) a single isolated first-order phase transition at $\lambda = \lambda_c = 1$, (ii) a single isolated continuous phase transition at $\lambda = \lambda_c = 1$ and (iii) two phase transitions with an intermediate massless phase (critical). While the last two possibilities imply that the Hamilton (3.2) is critical the first one implies it to be non-critical (finite mass gap). In the cases (i) and (ii) all the sequences for $q = 1, \ldots, \bar{N}$ in table 2 should converge to $\lambda = 1$ while in the last case the different sequences may converge to distinct points. Extrapolating the sequences of table 2 by using VBS approximants (Vanden Broeck and Schwartz 1979, Hamer and Barber 1981b) we obtain for $N = 5$

$$\lambda_c^{(1)} = 1.0000\,(0) \qquad \lambda_c^{((2)} = 1.0000\,(2)$$

**Table 2.** Sequences of estimators for the critical temperature of the Hamiltonians (4.1). $\lambda_c^{(q)}(L)$ are obtained using sectors 0 and $q$.

| $N$ | $L$ | $\lambda_c^{(1)}(L)$ | $\lambda_c^{(2)}(L)$ | $\lambda_c^{(3)}(L)$ | $\lambda_c^{(4)}(L)$ |
|---|---|---|---|---|---|
| 5 | 3 | 1.057 016 89 | 1.037 053 42 | | |
| | 4 | 1.021 479 45 | 1.011 860 43 | | |
| | 5 | 1.011 018 65 | 1.005 302 47 | | |
| | 6 | 1.006 643 96 | 1.002 847 23 | | |
| | 7 | 1.004 425 50 | 1.001 718 40 | | |
| | 8 | 1.003 153 07 | 1.001 125 07 | | |
| 6 | 3 | 1.059 592 21 | 1.039 617 00 | 1.033 997 63 | |
| | 4 | 1.022 670 02 | 1.013 245 08 | 1.010 474 50 | |
| | 5 | 1.011 720 06 | 1.006 186 56 | 1.004 521 93 | |
| | 6 | 1.007 107 49 | 1.003 462 32 | 1.002 349 72 | |
| | 7 | 1.004 754 15 | 1.002 171 19 | 1.001 374 79 | |
| 7 | 3 | 1.061 248 46 | 1.042 194 15 | 1.034 413 19 | |
| | 4 | 1.023 371 58 | 1.014 564 02 | 1.010 774 34 | |
| | 5 | 1.012 112 06 | 1.007 001 85 | 1.004 741 92 | |
| | 6 | 1.007 356 55 | 1.004 017 33 | 1.002 514 88 | |
| 8 | 3 | 1.062 297 04 | 1.044 394 24 | 1.035 822 74 | 1.033 178 70 |
| | 4 | 1.023 764 98 | 1.015 649 24 | 1.011 531 19 | 1.010 233 65 |
| | 5 | 1.012 314 52 | 1.007 658 12 | 1.005 222 27 | 1.004 446 25 |
| | 6 | 1.007 476 83 | 1.004 457 28 | 1.002 847 06 | 1.002 330 55 |

for $N = 6$

$$\lambda_c^{(1)} = 1.000\,00\,(7) \qquad \lambda_c^{(2)} = 1.0000\,(3) \qquad \lambda_c^{(3)} = 1.0000\,(1)$$

for $N = 7$

$$\lambda_c^{(1)} = 1.00\,(4) \qquad \lambda_c^{(2)} = 1.00\,(2) \qquad \lambda_c^{(3)} = 1.00\,(1)$$

and for $N = 8$

$$\lambda_c^{(1)} = 1.00\,(4) \qquad \lambda_c^{(2)} = 1.00\,(2) \qquad \lambda_c^{(3)} = 1.00\,(1) \qquad \lambda_c^{(4)} = 1.00\,(1)$$

where the errors are estimated to be in the last digit.

All these results strongly suggest that there is a single phase transition occurring at $\lambda = 1$. Although the nature of the phase transition cannot be determined from these results our subsequent analysis indicates the transition as being continuous.

The thermal exponents $\nu$ and $\alpha$ may be calculated using, respectively, the Callan-Symanzik $\beta$ functions (Hamer *et al* 1979)

$$\beta_L^{(q)}(\lambda) = -\Lambda_L^{(q)}(\lambda)/[\Lambda_L^q(\lambda) - 2\lambda\, \partial\Lambda_L^{(q)}/\partial\lambda] \qquad q = 1, 2, \ldots, \bar{N} \tag{4.7}$$

and the analogue of the specific heat per site

$$C_L(\lambda) = -(\lambda^2/L)\partial^2 E_0^{(0)}/\partial\lambda^2. \tag{4.8}$$

In table 3 we show, at $\lambda = 1$, the values of these functions together with the mass gaps

**Table 3.** Finite-size results. Listed are the values at $\lambda = \lambda_c = 1$ of the mass gap $\Lambda_L^{(q)}(q = 1, 2)$, the $\beta$ functions $\beta_L^{(q)}(q = 1, 2)$ and the specific heat $C_L(\lambda)$.

| Model | $L$ | $\Lambda_L^{(1)}(1)$ | $\Lambda_L^{(2)}(1)$ | $\beta_L^{(1)}(1)$ | $\beta_L^{(2)}(1)$ | $C_L(1)$ |
|---|---|---|---|---|---|---|
| $Z(5)$ | 2 | 2.222 415 179 | 2.951 764 728 | 0.302 927 480 | 0.316 275 389 | 1.727 546 4 |
| | 3 | 1.378 801 132 | 1.881 879 374 | 0.166 548 927 | 0.176 238 947 | 2.817 166 0 |
| | 4 | 1.001 817 610 | 1.388 298 182 | 0.110 328 601 | 0.117 553 553 | 3.766 043 8 |
| | 5 | 0.787 041 336 | 1.101 593 945 | 0.080 414 545 | 0.086 050 594 | 4.624 741 9 |
| | 6 | 0.648 098 818 | 0.913 634 271 | 0.062 166 443 | 0.066 724 026 | 5.472 479 8 |
| | 7 | 0.550 802 563 | 0.780 710 922 | 0.050 027 540 | 0.053 816 064 | 6.268 044 0 |
| | 8 | 0.478 856 835 | 0.681 666 394 | 0.041 451 272 | 0.044 668 541 | 7.037 061 3 |
| $Z(6)$ | 2 | 2.593 313 811 | 3.571 653 839 | 0.284 396 033 | 0.299 370 132 | 4.163 262 575 |
| | 3 | 1.588 128 837 | 2.257 399 520 | 0.151 696 014 | 0.162 365 585 | 6.973 865 635 |
| | 4 | 1.145 761 053 | 1.657 740 863 | 0.098 457 525 | 0.106 311 150 | 9.526 934 655 |
| | 5 | 0.895 958 630 | 1.311 569 479 | 0.070 656 683 | 0.076 721 942 | 11.960 015 675 |
| | 6 | 0.735 322 404 | 1.085 547 865 | 0.053 941 307 | 0.058 805 309 | 14.320 419 736 |
| | 7 | 0.623 329 524 | 0.926 176 842 | 0.042 952 641 | 0.046 967 063 | 16.630 874 396 |
| $Z(7)$ | 2 | 2.943 808 723 | 4.138 808 300 | 0.269 746 193 | 0.285 383 692 | 5.528 142 419 |
| | 3 | 1.782 168 270 | 2.592 493 295 | 0.140 341 537 | 0.151 293 105 | 9.466 494 771 |
| | 4 | 1.277 657 066 | 1.894 649 350 | 0.089 586 002 | 0.097 554 432 | 13.163 870 296 |
| | 5 | 0.994 949 554 | 1.494 323 454 | 0.063 488 434 | 0.069 588 196 | 16.773 466 174 |
| | 6 | 0.814 102 739 | 1.234 041 937 | 0.047 981 807 | 0.052 838 346 | 20.343 151 409 |
| | 7 | 0.688 506 257 | 1.051 077 050 | 0.037 885 337 | 0.041 869 175 | 23.893 769 082 |
| $Z(8)$ | 2 | 3.279 127 855 | 4.669 799 466 | 0.257 793 106 | 0.273 626 975 | 7.018 675 684 |
| | 3 | 1.965 022 528 | 2.899 932 628 | 0.131 330 422 | 0.142 246 634 | 12.244 308 128 |
| | 4 | 1.400 841 721 | 2.109 386 229 | 0.082 675 147 | 0.090 535 963 | 17.282 908 308 |
| | 5 | 1.086 823 857 | 1.658 580 667 | 0.057 982 107 | 0.063 952 680 | 22.298 497 521 |
| | 6 | 0.886 869 244 | 1.366 657 761 | 0.043 455 130 | 0.048 178 956 | 27.335 907 312 |
| | 7 | 0.748 474 941 | 1.162 057 342 | 0.034 072 258 | 0.037 926 759 | 32.411 371 974 |

(4.6). Although we have calculated these functions for $q = 1, 2, \ldots, \bar{N}$ we present, for brevity, only the $q = 1, 2$ results. From FSS theory we expect (Barber 1983) that as $L \to \infty$

$$\beta_L^{(q)}(\lambda_c) \sim L^{1/\nu} \qquad \text{and} \qquad C_L(\lambda_c) \sim L^{\alpha/\nu}. \tag{4.9}$$

Extrapolating the sequences (4.7) and (4.8) by using the alternate $\varepsilon$ algorithm (Hamer and Barber 1981b) we obtain the values

$$\begin{array}{lll} N = 5 & 1/\nu = 1.415 \pm 0.005 & \alpha/\nu = 0.82 \pm 0.02 \\ N = 6 & 1/\nu = 1.49 \pm 0.02 & \alpha/\nu = 0.94 \pm 0.02 \\ N = 7 & 1/\nu = 1.54 \pm 0.02 & \alpha/\nu = 1.04 \pm 0.02 \\ N = 8 & 1/\nu = 1.59 \pm 0.02 & \alpha/\nu = 1.10 \pm 0.05 \end{array}$$

which gives us the following values for the anomalous dimension $X_\varepsilon = 2 - 1/\nu$ of the energy operator:

$$\begin{array}{llll} N = 5 & X_\varepsilon = 0.585 \pm 0.05 & N = 6 & X_\varepsilon = 0.51 \pm 0.02 \\ N = 7 & X_\varepsilon = 0.46 \pm 0.02 & N = 8 & X_\varepsilon = 0.41 \pm 0.02. \end{array}$$

We thus verify that the values of $X_\varepsilon$ agree reasonably well with the predicted values given by (1.3). The estimates of $\alpha/\nu$ are however slightly lower for $N > 5$ than the predicted values $2(1 - X_\varepsilon)$, which we attribute to the small number of terms in the extrapolating sequence and the fact that normally the convergence of the specific heat VBS approximants are relatively slow (Hamer and Barber 1981b, Alcaraz and Drugowich de Felício 1984).

### 4.2. *Conformal invariance and mass gap amplitudes*

Most of the statistical mechanical systems at criticality (Cardy 1987) are believed to satisfy the basic assumptions—short-range interactions, scale invariance, rotational and translation invariance—that ensure conformal invariance (Polyakov 1970). In two dimensions this symmetry has many important implications (see Cardy (1987) for a recent review). Specifically, Cardy (1984, 1986) has derived a set of remarkable relations between the eigenvalue spectrum of the transfer matrix in a strip of finite-size width and the anomalous dimension of the operators describing the critical behaviour of the infinite system. The results for the quantum Hamiltonian formalism, which we are interested in, can be transcribed as follows. To each primary operator $\varphi$ with anomalous dimension $X_\varphi$ and spin $s_\varphi$ in the operator algebra of the infinite system there exists an infinite set of states in the quantum Hamiltonian, in a periodic chain of $L$ sites, whose energy and momentum as $L \to \infty$, at $\lambda = \lambda_c$, is given by

$$E_{n,n'} = E_0^{(0)} + (2\pi/L)\zeta(X_\varphi + n + n') + o(L^{-1}) \qquad n, n' = 0, 1, 2, \ldots \tag{4.10a}$$

$$P_{n,n'} = (2\pi/L)(s_\varphi + n - n') \qquad n, n' = 0, 1, 2, \ldots \tag{4.10b}$$

respectively. The constant $\zeta$ does not appear in the transfer matrix formalism but enters in the Hamiltonian relations since the Hamiltonian may in principle be multiplied by an arbitrary constant without modifying its critical properties (see for example Alcaraz and Drugowich de Felício 1984, von Gehlen *et al* 1986, Alcaraz and Barber 1987a, b).

Before we proceed further in the application of the above relations let us introduce the following notation for the eigenenergies of our Hamiltonian (4.1) at the critical temperature $\lambda = \lambda_c = 1$. We denote by $E_n^{(q)}(k)$ the energy corresponding to the $n$-excited state in the sector with charge $Q = q$ and momentum $k$. The $Z(N)$ neutral operators are related to states in the $q = 0$ sector, while the order and disorder operators are related to the states in the charged sectors ($q = 1, 2, \ldots, N-1$). The first neutral operator is the energy operator whose anomalous dimension $X_\varepsilon$ may be estimated through

$$G_L^{(0)}(1) \equiv E_1^{(0)}(0) - E_0^{(0)}(0) = \frac{2\pi\zeta}{L} X_\varepsilon + o(L^{-1}) \tag{4.11}$$

while the anomalous dimension $X_{\varepsilon\varepsilon}$ for the second neutral operator through

$$G_L^{(0)}(2) \equiv E_2^{(0)}(0) - E_0^{(0)}(0) = \frac{2\pi\zeta}{L} X_{\varepsilon\varepsilon} + o(L^{-1}). \tag{4.12}$$

The other neutral operators for $5 < N < 9$ have dimensions higher than two and are thus irrelevant. The estimation of their dimensions is rather difficult because, apart from being related to higher states in the spectrum, these states may also be confused

**Table 4.** Ratio of mass gap amplitudes. See equations (4.11)-(4.14).

| Model | $L$ | $G_L^{(0)}(1)/Z_L$ | $G_L^{(0)}(2)/Z_L$ | $G_L^{(1)}(1)/Z_L$ | $G_L^{(2)}(1)/Z_L$ | $G_L^{(3)}(1)/Z_L$ | $G_L^{(4)}(1)/Z_L$ |
|---|---|---|---|---|---|---|---|
| $Z(5)$ | 2 | 1.172 6680 | 1.901 6161 | 0.209 7130 | 0.278 5364 | | |
| | 3 | 0.881 0805 | 1.714 6664 | 0.152 8418 | 0.208 6086 | | |
| | 4 | 0.787 1071 | 1.695 1805 | 0.137 0488 | 0.189 9194 | | |
| | 5 | 0.740 9454 | 1.699 4783 | 0.130 1189 | 0.182 1228 | | |
| | 6 | 0.713 3835 | 1.707 3789 | 0.126 3577 | 0.178 1283 | | |
| | 7 | 0.694 9456 | 1.714 8301 | 0.124 0421 | 0.175 8181 | | |
| | 8 | 0.681 6602 | 1.721 0695 | 0.122 4911 | 0.174 3697 | | |
| | 9 | 0.671 5755 | 1.726 1138 | 0.121 3876 | 0.173 4077 | | |
| $Z(6)$ | 2 | 1.124 7660 | 1.836 3205 | 0.200 3420 | 0.275 9219 | 0.297 8765 | |
| | 3 | 0.833 2422 | 1.642 4296 | 0.144 8520 | 0.205 8957 | 0.224 1560 | |
| | 4 | 0.737 9070 | 1.611 5143 | 0.129 2374 | 0.186 9868 | 0.204 5476 | |
| | 5 | 0.690 5021 | 1.605 5527 | 0.122 2850 | 0.179 0097 | 0.196 4465 | |
| | 6 | 0.661 8907 | 1.604 7652 | 0.118 4528 | 0.174 8705 | 0.192 3488 | |
| | 7 | 0.642 5662 | 1.604 9020 | 0.116 0560 | 0.172 4424 | 0.190 0154 | |
| | 8 | 0.628 5230 | 1.604 9444 | 0.144 4255 | 0.170 8961 | 0.188 5791 | |
| $Z(7)$ | 2 | 1.086 0285 | 1.783 8622 | 0.191 9833 | 0.269 9163 | 0.302 8397 | |
| | 3 | 0.795 3422 | 1.584 2493 | 0.137 8136 | 0.200 4753 | 0.227 8793 | |
| | 4 | 0.699 3171 | 1.544 2046 | 0.122 3982 | 0.181 5053 | 0.207 8607 | |
| | 5 | 0.651 1801 | 1.530 2263 | 0.115 4505 | 0.173 3961 | 0.199 5664 | |
| | 6 | 0.621 9196 | 1.522 7827 | 0.111 5723 | 0.169 1247 | 0.195 3591 | |
| | 7 | 0.602 0329 | 1.517 4109 | 0.109 1161 | 0.166 5772 | 0.192 9562 | |
| $Z(8)$ | 2 | 1.053 9003 | | 0.184 6683 | 0.262 9858 | 0.301 8189 | 0.313 8423 |
| | 3 | 0.764 3879 | | 0.131 7068 | 0.194 3697 | 0.226 6437 | 0.236 7820 |
| | 4 | 0.668 0525 | | 0.116 4919 | 0.175 4134 | 0.206 4131 | 0.216 2312 |
| | 5 | 0.619 4857 | | 0.109 5655 | 0.167 2057 | 0.197 9611 | 0.207 7550 |
| | 6 | 0.589 8187 | | 0.105 6595 | 0.162 8203 | 0.193 6322 | 0.203 4830 |
| | 7 | 0.569 5690 | | 0.103 1608 | 0.160 1641 | 0.191 1321 | 0.201 0630 |

**Table 5.** Estimated values for the anomalous dimension of the neutral ($X_\varepsilon$ and $X_{\varepsilon\varepsilon}$) and the $Z(N)$ charged operators ($X_{\rm m}^{(q)}$, $q = 1, 2, \ldots, N-1$). The conjectured values are given by (1.2) and (1.3).

| | | $N=5$ | $N=6$ | $N=7$ | $N=8$ |
|---|---|---|---|---|---|
| $X_\varepsilon$ | Extrapolated | $0.572 \pm 0.002$ | $0.503 \pm 0.004$ | $0.45 \pm 0.01$ | $0.41 \pm 0.02$ |
| | Conjectured | 0.571 42 ... | 0.50 | 0.444 4 ... | 0.40 |
| $X_{\varepsilon\varepsilon}$ | Extrapolated | $1.73 \pm 0.02$ | $1.55 \pm 0.05$ | — | — |
| | Conjectured | 1.714 28 ... | 1.50 | 1.333 3 ... | 1.20 |
| $X_{\rm m}^{(1)}$ | Extrapolated | $0.1143 \pm 0.001$ | $0.1042 \pm 0.0002$ | $0.097 \pm 0.001$ | $0.090 \pm 0.001$ |
| | Conjectured | 0.114 285 ... | 0.104 166 ... | 0.095 23 ... | 0.087 5 |
| $X_{\rm m}^{(2)}$ | Extrapolated | $0.1712 \pm 0.0001$ | $0.1662 \pm 0.0005$ | $0.158 \pm 0.001$ | $0.151 \pm 0.001$ |
| | Conjectured | 0.171 428 ... | 0.166 66 ... | 0.158 73 ... | 0.150 |
| $X_{\rm m}^{(3)}$ | Extrapolated | $0.1712 \pm 0.0001$ | $0.1863 \pm 0.0005$ | $0.188 \pm 0.002$ | $0.186 \pm 0.002$ |
| | Conjectured | 0.171 428 ... | 0.187 5 | 0.190 476 ... | 0.187 5 |
| $X_{\rm m}^{(4)}$ | Extrapolated | $0.1143 \pm 0.0001$ | $0.1662 \pm 0.0005$ | $0.188 \pm 0.002$ | $0.198 \pm 0.002$ |
| | Conjectured | 0.114 285 ... | 0.166 66 ... | 0.190 476 ... | 0.20 |

with the higher states ($n = n' = 1$ in (4.10)) related to the energy operator. The charged $q$ operators, with dimension $X_{\rm m}^{(q)}(q = 1, 2, \ldots, N-1)$, are those governing the long distance correlations

$$\langle S^q(i) S^{+q}(i+n)\rangle \xrightarrow[n\to\infty]{} |n|^{-2x_m^{(q)}} \qquad q = 1, 2, \ldots, N-1.$$

Their dimensions can be estimated by the relations

$$G_L^{(q)}(1) \equiv E_0^{(q)}(0) - E_0^{(0)}(0) = \frac{2\pi\zeta}{L} X_{\rm m}^{(q)} + \mathrm{o}(L^{-1}) \qquad q = 1, 2, \ldots, N-1. \tag{4.13}$$

The degeneracy, already mentioned, of the disjoint sectors of the Hilbert space with charge $q$ and $N-q$ implies that $X_{\rm m}^{(q)} = X_{\rm m}^{(N-q)}$, $q = 1, \ldots, N-1$, in perfect agreement with the prediction (1.2). The constant $\zeta$ appearing in the preceding equations can be extracted from the difference in energy of two successive states related to the same primary operator, for example using the charge $q = 1$ operator we have

$$Z_L \equiv E_0^{(1)}\left(\frac{2\pi}{L}\right) - E_0^{(1)}(0) = \frac{2\pi\zeta}{L} + \mathrm{o}(L^{-1}). \tag{4.14}$$

In table 4 we present our estimators (4.11)-(4.14) for $N = 5$-8, respectively. The extrapolation of these sequences, using the alternate $\varepsilon$ algorithm (Hamer and Barber 1981b) gives the values shown in table 5. For the sake of comparison we have also presented in this table the conjectured values given by (1.2) and (1.3). As we can see the agreement is good for all $N$, which strongly supports the conjecture that the $Z(N)$ quantum field theories constructed by Zamolodchikov and Fateev (1985) are the underlying field theories of these critical statistical systems (3.2).

## 5. Conformal anomaly

In this section we estimate the conformal anomaly or central charge of the Virasoro algebra governing the critical behaviour of the Hamiltonian (3.2) and the models given

**Table 6.** Finite-size sequences and extrapolated values for the conformal anomaly $c$. The conjectured values are given by (1.1).

| $L$ | $-E_0^{(0)}(0)L$ | $-12(E_0^{(0)}(0)-e_0L)/Z_L$ | $-12(E_0^{(0)}(0)-e_0L)L/10\pi$ | $-12(E_0^{(0)}(0)-e_0L)L/12\pi$ | $-12(E_0^{(0)}(0)-e_0L)L/14\pi$ | $-12(E_0^{(0)}(0)-e_0L)L/16\pi$ |
|---|---|---|---|---|---|---|
| 2 | 7.265 084 54 | 1.888 887 | 1.274 342 | | | |
| 3 | 6.781 381 37 | 1.398 128 | 1.204 420 | | | |
| 4 | 6.624 093 43 | 1.267 735 | 1.179 918 | | | |
| 5 | 6.553 364 83 | 1.213 515 | 1.168 214 | | | |
| 6 | 6.515 504 25 | 1.185 820 | 1.161 608 | | | |
| 7 | 6.492 871 28 | 1.169 858 | 1.157 465 | | | |
| 8 | 6.478 262 83 | 1.159 890 | 1.154 670 | | | |
| 9 | 6.468 285 41 | 1.153 301 | 1.152 681 | | | |
| Extrapolated | 6.431 029 7... | 1.13 (5) | 1.142 (9) | | | |
| Conjectured | | 1.142 857... | 1.142 857... | | | |
| 2 | 9.546 724 13 | 2.054 696 | | 1.411 009 | | |
| 3 | 8.902 104 28 | 1.522 192 | | 1.328 070 | | |
| 4 | 8.693 479 29 | 1.380 400 | | 1.298 495 | | |
| 5 | 8.599 887 61 | 1.321 450 | | 1.284 116 | | |
| 6 | 8.549 861 06 | 1.291 375 | | 1.275 870 | | |
| 7 | 8.519 984 75 | 1.274 075 | | 1.270 610 | | |
| 8 | 8.500 714 94 | 1.263 302 | | 1.267 016 | | |
| Extrapolated | 8.438 520 78... | 1.24 (5) | | 1.250 (1) | | |
| Conjectured | | 1.250 000... | | 1.250 000... | | |
| 2 | 11.944 992 87 | 2.184 167 | | | 1.522 945 | |
| 3 | 11.131 104 78 | 1.619 039 | | | 1.428 097 | |
| 4 | 10.868 800 32 | 1.468 154 | | | 1.393 778 | |
| 5 | 10.751 378 47 | 1.405 365 | | | 1.376 851 | |
| 6 | 10.688 697 28 | 1.373 322 | | | 1.367 001 | |
| 7 | 10.651 297 67 | 1.354 893 | | | 1.360 650 | |
| Extrapolated | 10.549 521 77... | 1.32 (4) | | | 1.333 (5) | |
| Conjectured | | 1.333 333... | | | 1.333 333... | |
| 2 | 14.443 280 21 | 2.288 876 | | | | 1.617 143 |
| 3 | 13.453 226 53 | 1.697 283 | | | | 1.511 351 |
| 4 | 13.135 338 97 | 1.538 871 | | | | 1.472 604 |
| 5 | 12.993 309 00 | 1.472 850 | | | | 1.453 264 |
| 6 | 12.917 583 06 | 1.439 121 | | | | 1.441 888 |
| 7 | 12.872 437 95 | 1.419 704 | | | | 1.434 458 |
| Extrapolated | 12.749 812 42... | 1.3 (8) | | | | 1.400 (3) |
| Conjectured | | 1.400 000... | | | | 1.400 000... |

by (2.2) and (3.1). For $N=2$ and 3 these models reduce to the critical Ising and three-state Potts models whose conformal class has a central charge $c=\frac{1}{2}$ and $c=\frac{4}{5}$ respectively (Belavin *et al* 1984a, b) in perfect agreement with the prediction (1.1). For $N=4$ the model reduces to a special point (see also § 4) of the critical Ashkin–Teller model, being governed by the same conformal class, with central charge $c=1$ (von Gehlen and Rittenberg 1987), that governs the eight-vertex and *XXZ* models (Blöte *et al* 1986). In the rest of this section we will extract numerically the values of $c$ for $5 \leq N < 9$.

The assumption of conformal invariance, at criticality, of the infinite (1+1) statistical systems has other implications, beyond (4.10), for the finite system. The ground-state energy for a finite Hamiltonian of size $L$ and periodic boundary conditions, at the critical point, should behave as (Blöte *et al* 1986, Affleck 1986)

$$E_0^{(0)}(0)/L = e_0 - \tfrac{1}{6}\pi c \zeta / L^2 + o(L^{-2}) \qquad L \to \infty \tag{5.1}$$

where $c$, as before, is the conformal anomaly of the conformal class governing the criticality of the infinite system, $e_0$ is the bulk limit of the ground-state energy per particle and $\zeta$ is the same constant that appears in (4.11)–(4.13). The values of $e_0$ for the Hamiltonians (3.2) are given exactly by (3.5) and these values are shown in table 1 for $N<9$. From (5.1) and (4.14) a possible way to extract $c$ is by extrapolating the sequence

$$C_L = -12(E_0^{(0)}(0) - Le_0)/Z_L. \tag{5.2}$$

In table 6 we exhibit these sequences for $N=5$–8 respectively. We also show in these tables their extrapolated values. As we can see they are in reasonable agreement with the predictions (1.1). One of the major error sources in these estimates concerns the evaluation of the constant $\zeta$. From the Ising exact solution $\zeta=2$ for $N=2$ and earlier finite-size calculations suggest $\zeta=3$ for $N=3$ (von Gehlen *et al* 1986) and $\zeta=4$ for $N=4$ (Alcaraz and Drugowich de Felício 1984). This indicates the conjecture of $\zeta=N$ for all the Hamiltonians (3.2) (Alcaraz 1986). This constant may be estimated by using the following sequence

$$\zeta_L(N) \equiv (LZ_L)/2\pi \tag{5.3}$$

where $Z_L$ is given by (4.14). In table 7 we present these sequences for $N=5$–8. We see from this table that $\zeta_L(N)$ exceeds the conjectured value for $N>5$. However the extrapolations of the above sequence either using VBS approximants or three-point fits (Alcaraz and Barber 1987a) are not conclusive, which may be explained if the sequence

**Table 7.** Finite-size sequence $\zeta_L(N)$ for $N=5$–9 (see equation (5.3)).

| $L$ | $\zeta_L(5)/10\pi$ | $\zeta_L(6)/12\pi$ | $\zeta_L(7)/14\pi$ | $\zeta_L(8)/16\pi$ |
|---|---|---|---|---|
| 2 | 0.674 652 | 0.686 724 | 0.697 266 | 0.706 523 |
| 3 | 0.861 452 | 0.872 472 | 0.882 064 | 0.890 453 |
| 4 | 0.930 729 | 0.940 664 | 0.949 339 | 0.956 937 |
| 5 | 0.962 670 | 0.971 748 | 0.979 710 | 0.986 701 |
| 6 | 0.979 582 | 0.987 990 | 0.995 397 | 1.001 919 |
| 7 | 0.989 407 | 0.997 279 | 1.004 244 | 1.010 394 |
| 8 | 0.995 500 | 1.002 934 | | |
| 9 | 0.999 461 | | | |

(5.3) did not reach its asymptotic regime. Our main argument in favour of the conjecture $\zeta = N$ concerns the last columns of table 6 where this value was used to extract $c$. As we clearly see the rate of convergence of these sequences is increased and the agreement with the conjectured values (1.1) is excellent.

## 6. General $Z(5)$ model—the bifurcation point

In this section we wish to test the conjecture mentioned in § 1, namely that for $N \geqslant 5$ the Boltzmann weights given by (3.1) correspond to the bifurcation point in the phase diagram of the general model (2.2) where a massless phase occurs (see points $E_1$ and $E_2$ in figure 1). Our analysis will be restricted to the $Z(5)$ model.

Instead of working with the general $Z(5)$ Euclidean model given by (2.2) we work here with its quantum Hamiltonian analogue described by

$$-H_5 = \sum_i \{(S(i)S^+(i) + \text{HC}) + \Omega((S(i)S^+(i))^2 + \text{HC}) + \lambda[C_1(R(i) + R^+(i)) + \Omega(R^2(i) + R^{+2}(i))]\} \tag{6.1}$$

where as before $\lambda$ plays the role of the temperature and $\Omega$ is an additional coupling constant. This Hamiltonian as well as (2.2) is self-dual. The self-dual line is given by $\lambda = 1$ $(\forall \Omega)$. It is interesting to remark upon three special cases of this general Hamiltonian: (i) $\Omega = 1$: it reduces to the five-state quantum Potts model which has a unique first-order phase transition at $\lambda = 1$ (Hamer 1981), (ii) $\Omega = 0$: it reduces to the $Z(5)$ quantum clock model which is believed as already having an intermediate massless phase separating two infinite-order phase transitions (Hamer and Barber 1981b, Alcaraz and Köberle 1980, 1981) and (iii) $\Omega = \Omega_0 = \sin(\pi/5)/\sin(2\pi/5) \approx 6.18$: it gives the Hamiltonian (4.1), the critical point of which is conjectured (Fateev and Zamolodchikov 1982) as being the bifurcation point in the phase diagram of (6.1).

We estimate the critical temperature of (6.1) for several values of $\Omega$ by using the sequences $\lambda_c^{(q)}(L)$, $q = 1, 2$; $L = 3$-$8$ obtained by solving (4.6) (see also § 4.1). In table 8 we present some of these sequences with their VBS extrapolated values. The results shown in this table together with those for $\Omega = \Omega_0$ given in table 2 indicate that the bifurcation point occurs very close to $\Omega_0$. This is clearly consistent with the conjecture that for $N \geqslant 5$ the quantum Hamiltonian (3.2) and the Euclidean model at (3.1) correspond to the bifurcation point of the general $Z(N)$ model.

**Table 8.** Sequences of estimators for the critical temperature of the Hamiltonian (6.1). $\lambda_c^{(q)}(L)$ are obtained by using sectors 0 and $q$.

| | $\Omega = 0.50$ | | $\Omega = 0.55$ | | $\Omega = 0.58$ | |
|---|---|---|---|---|---|---|
| $L$ | $\lambda_c^{(1)}(L)$ | $\lambda_c^{(2)}(L)$ | $\lambda_c^{(1)}(L)$ | $\lambda_c^{(2)}(L)$ | $\lambda_c^{(1)}(L)$ | $\lambda_c^{(2)}(L)$ |
| 3 | 1.059 813 | 1.029 422 | 1.058 546 | 1.033 003 | 1.057 850 | 1.034 895 |
| 4 | 1.022 420 | 1.006 509 | 1.021 967 | 1.009 089 | 1.021 738 | 1.010 405 |
| 5 | 1.011 437 | 1.001 130 | 1.011 212 | 1.003 178 | 1.011 115 | 1.004 198 |
| 6 | 1.006 868 | 0.999 416 | 1.006 729 | 1.001 123 | 1.006 681 | 1.001 958 |
| 7 | 1.004 565 | 0.998 801 | 1.004 465 | 1.000 268 | 1.004 437 | 1.000 975 |
| 8 | 1.003 253 | 0.998 584 | 1.003 171 | 0.999 874 | 1.003 153 | 1.000 487 |
| Extrapolated | 1.0000 (8) | 0.9984 (3) | 1.0000 (3) | 0.9994 (2) | 1.0000 (2) | 0.9999 (3) |

## 7. Summary and conclusions

Our aim in this paper was to study the critical behaviour of a family of self-dual $Z(N)$ models which are exactly soluble at a particular point (Fateev and Zamolodchikov 1982). These points correspond, in the parameter space of the general self-dual $Z(N)$ model (2.2), to the points where the Boltzmann weights are given by (3.1). Two conjectures exist regarding this family of exactly solved points. The first of these asserts that they are critical points having their critical behaviour governed by a recently constructed $Z(N)$ invariant quantum field theory (Zamolodchikov and Fateev 1985). In this case the critical indices as well as the conformal anomaly corresponding to these points are predicted by (1.1)-(1.3). The second conjecture states that this family of points for $N \geq 5$ correspond to the bifurcation points in the parameter space of (2.2) where a massless phase occurs (points $E_1$ and $E_2$ in figure 1 for $N \geq 5$).

Instead of working directly with the Euclidean models (2.3) and (3.1) we study the family of quantum Hamiltonians (3.2). These Hamiltonians commute with the diagonal-to-diagonal transfer matrices associated with these exactly soluble points. Their ground-state energy, in the bulk limit, is also known exactly (Alcaraz and Lima Santos 1986). Our analysis, which can be divided into two parts, was performed by mainly studying the finite-size behaviour of the Hamiltonians ($N < 9$) with periodic boundary conditions applied. We firstly calculated the critical temperature and the thermal exponents $\nu$ and $\alpha$ (see § 4.1) by using standard FSS theory (Barber 1983). Secondly we exploited recent predictions of conformal invariance (Cardy 1987) concerning the eigenvalue spectrum of statistical systems on a strip of finite width (see §§ 4.2 and 5.4). We were able to obtain the anomalous dimensions of the energy operators ($Z(N)$ neutral) as well as the dimensions of all the $Z(N)$ charged (order and disorder) operators. We also estimate the conformal anomaly $c$ corresponding to the conformal theory which governs these statistical models. All our results in §§ 4 and 5 strongly support the conjecture that the family of models (3.2) (or (2.2) and (3.1)) are critical, having their behaviour ruled by the $Z(N)$ quantum field theory of Zamolodchikov and Fateev (1985).

Finally concerning the possibility of these statistical systems being related to the bifurcation points, mentioned earlier, our results (see § 6) for $N = 5$, although not fully guaranteeing the validity of this conjecture, are clearly consistent with it.


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